Associating Finite Groups to Dessins D' Enfants

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Objective

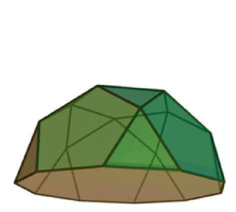
- **Overarching Research Interest**:
  Given a loopless, connected, planar, bipartite graph Γ, use properties of the associated symmetry group G to construct a Belyi map $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ such that Γ arises as its Dessin D'Enfants
- **Current Research Project**:
  Finding Belyi maps in order to realize all the Johnson solids as Dessins d'Enfants.

Background

1. In 1984, Alexander Grothendieck, inspired by a result of Gennadii Belyi from 1979, constructed a finite, connected planar graph via certain rational functions by looking at the inverse image of the interval from 0 to 1. This gave rise to **Grothendieck's theory of Dessin D' Enfants**. Each conceivable Dessin D'Enfants Δ_β could be realized by some **Belyi Map** $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$.
2. It was known to Felix Klein that every Platonic solid could be realized as the Dessin of a Belyi map β.
3. Archimedean and Catalan solids are duals of each other. They are derived from the platonic solids via 8 'operations'. In 2001, N. Magot and A. Zvonkin showed that the Archimedean and Catalan solids can be realized as Dessins d'Enfants[1]. This was done by finding the operations' Belyi maps ϕ and writing down factorizations $\beta' = \phi \circ \beta$ for each Archimedean solid. The corresponding Catalan solid has the Belyi map $1/\beta'$.
4. Most of the 92 **Johnson solids** can be derived via 6 'operations' on the *Platonic solids, Archimedean and Catalan solids, Prisms and Antiprisms,Cupolae, Pyramids, Rotunda.* Theses operations are:
   - *Bi*: join two copies of a solid base-to-base.
   - *Elongate*: adding a prism to a solid's base.
   - *Gyroelongate*: adding an antiprism to a solid's base.
   - *Augment*: adding a pyramid/cupola to the solid.
   - *Diminish*: removing a pyramid/cupola from the solid.
   - *Gyrate*: rotate a cupola on the solid so that different edges match up.
5. Let $\mathbb{P} = \mathbb{P}^1(\mathbb{C})$ be the complex projective line. A **Belyi Map** is a function $\beta : \mathbb{P} \to \mathbb{P}$ that is rational and unramified outside $\{0, 1, \infty\}$. We denote $G \subset \mathrm{Aut}(\mathbb{P})$ the group of Mobius transformations satisfying $\beta \cdot \gamma(z) = \beta(z)$ for all $z \in \mathbb{P}$. Felix Klien showed the existence of non-trivial β with non-trivial G.
6. Given a Belyi map $\beta : \mathbb{P} \to \mathbb{P}$, the **Dessin d'Enfants** Δ_β associated to β is a connected, bipartite, planar graph defined as follows:
   - the set of "black" coloured vertices are $B = \beta^{-1}(0)$,
   - the set "white" coloured vertices are $W = \beta^{-1}(1)$,
   - the edges are given by $E = \beta^{-1}([0, 1])$.
   - The midpoints of the faces are $F = \beta^{-1}(\infty)$.

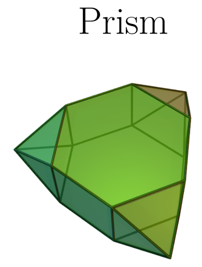
   The graph Δ_β has symmetry reflected by G.
7. A **Johnson Solid** is a convex polyhedron with regular polygons as faces but which is not a Platonic or Archimedean [3]. All 92 Johnson solids have either Cyclic symmetry or Dihedral symmetry.

Pentagonal Rotunda

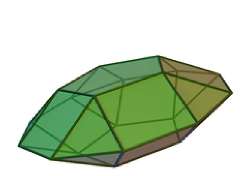


$Z_n = \langle r | r^n = 1 \rangle$

Triaugmented Hexagonal Prism



Pentagonal Bicopula



$D_n = \langle r, s | s^2 = r^n = (sr)^2 = 1 \rangle$

Mathematical Section

- **Theorem** *(Felix Klein)* : if G is a finite subgroup of $\mathrm{Aut}(\mathbb{P})$, then G is isomorphic to one of the five types of groups: Z_n, D_n, A_4, A_5, S_4.
- **Theorem** *(Riemann's Existence Theorem)* : For any hypermap, the corresponding Belyi function exists and is unique to a linear fractional transfromation of the variable z.
- We represent a given solid as a hypermap and, subsequently, as a reduced hypermap via the associated symmetry group G.
- We compute the Belyi function β' of the reduced hypermap[2]:

1. The 'Black' vertices must be the roots of the function β'. The multiplicities of each root must match the corresponding vertex's order. Consider a hypermap Γ with $|B| = 5$ vertices with degrees $\{3, 3, 2, 2, 2\}$, the numerator of β' can be written in the form $(w^2 + aw + b)^3\,(w^3 + cw^2 + dw + e)^2$
2. Assume Γ has $|E| = 6$ edges. Then, there are 6 'White' vertices each of degree 2. These vertices must be the roots of the function $\rho = \beta' - 1$. The numerator of ρ can be written in the form $(w^6 + mw^5 + nw^4 + pw^3 + qw^2 + rw)^2$.
3. Assume Γ has $|F| = 3$ faces with valencies $\{5, 4, 3\}$. Then the denomitnator of β' factorizes as $(w - K)^5(w - J)^4(w - L)^3$.

We end up with:

$$\beta'(w) = W \cdot \frac{(w^2 + aw + b)^3(w^3 + cw^2 + dw + e)^2}{(w - K)^5(w - J)^4(w - L)^3}$$

,

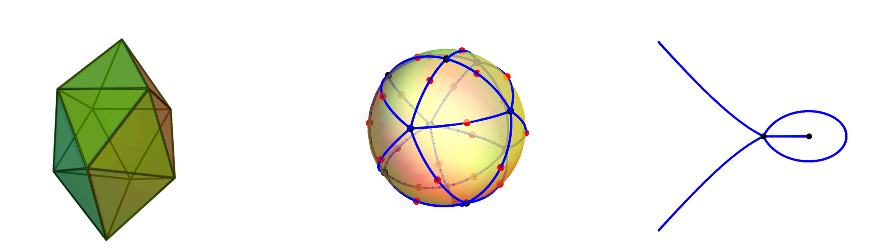
$$\beta'(w) - 1 = W \cdot \frac{(w^6 + mw^5 + nw^4 + pw^3 + qw^2 + rw)^2}{(w - K)^5(w - J)^4(w - L)^3}$$

The coefficients a, b, c, d, e, m, n, p, q, r, K, J, L, W are determined by using mathematical software packages. For our project, we used Mathematica 9.

- Finally, we deduce the Belyi map β of the non-reduced hypermap :

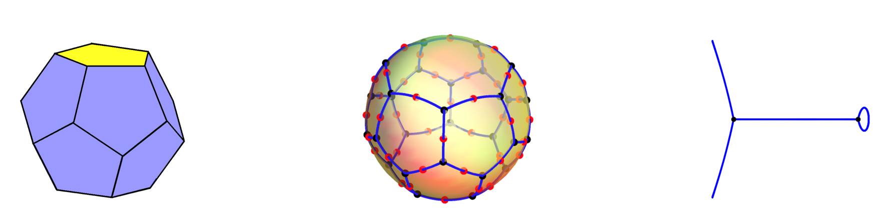
1. If the solid has cyclic symmetry, we find a Belyi map $\phi(z) = \dfrac{sz + t}{uz + v}$. If the solid has dihedral symmetry, we find the Belyi map $\phi(z) = \dfrac{s(z^n - 1)^2 - 4tz^n}{u(z^n - 1)^2 - 4vz^n}$. The coefficients s, t, u, v are also determined with mathematical software packages.
2. We write $\beta(z) = \beta'(\phi(z))$.

Results



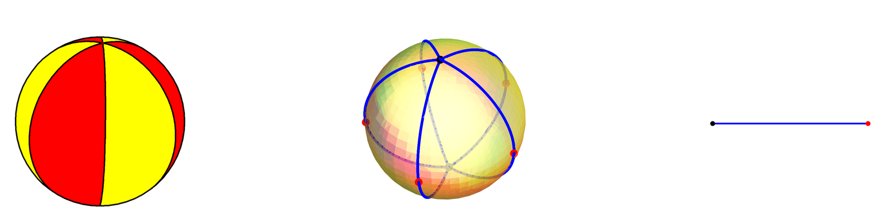
- Gyroelongated Bipyramids
- Rotation Group D_n
- $\beta(z) = \dfrac{1728\,\zeta_4\,z^n\,(z^{2n} - 11\,\zeta_4\,z^n + 1)^5}{(z^{4n} + 228\,\zeta_4\,z^{3n} - 494\,z^{2n} + 228\,\zeta_4\,z^n + 1)^3}$

- $\beta'(w) = \dfrac{1728\,w^5\,(w - 1)}{25\,(11 + 18G)\,(4w - G^3)^3}$
- $\phi(z) = \dfrac{z^{2n} - 11\,\zeta_4\,z^n + 1}{z^{2n} + 4\,\zeta_4\,(41 - 25G)\,z^n + 1}$



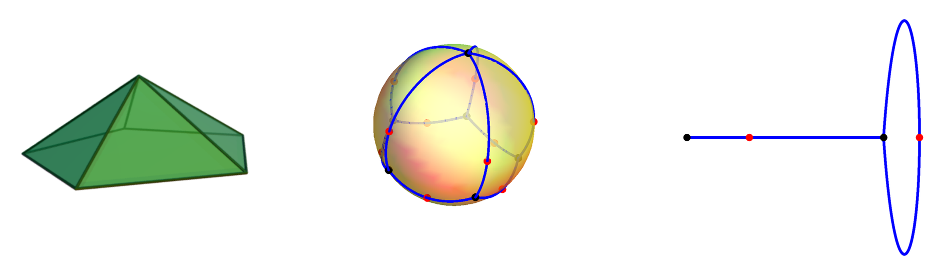
- Truncated Trapezohedra
- Rotation Group D_n
- $\beta(z) = \dfrac{(z^{4n} + 228\,\zeta_4\,z^{3n} - 494\,z^{2n} + 228\,\zeta_4\,z^n + 1)^3}{1728\,\zeta_4\,z^n\,(z^{2n} - 11\,\zeta_4\,z^n + 1)^5}$

- $\beta'(w) = \dfrac{25\,(11 + 18G)\,w^3(G^3\,w - 4)^3}{1728(w - 1)}$
- $\phi(z) = \dfrac{z^{2n} + 4\zeta_4\,(41 - 25G)\,z^n + 1}{z^{2n} - 11\,\zeta_4\,z^n + 1}$



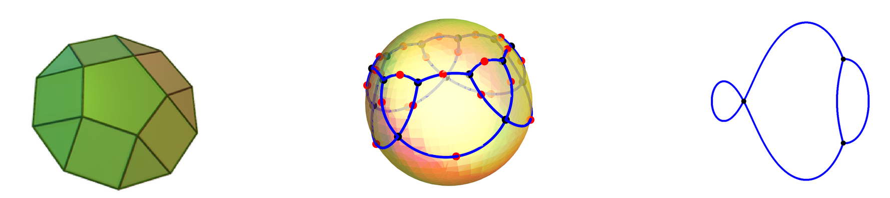
- Dipoles/ Hosohedron
- Rotation Group D_n
- $\beta(z) = -\dfrac{(z^n - 1)^2}{4\,z^n}$

- $\beta'(w) = w$
- $\phi(z) = -\dfrac{(z^n - 1)^2}{4\,z^n}$



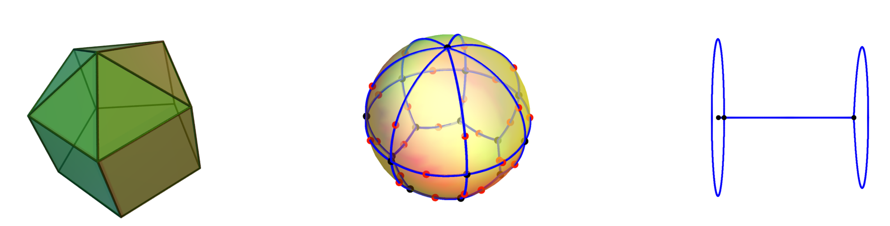
- Wheels/Pyramids
- Rotation Group Z_n
- $\beta(z) = \dfrac{z^n\,(z^n + 8)^3}{64\,(z^n - 1)^3}$

- $\beta'(w) = \dfrac{w^3\,(w + 8)}{64\,(w - 1)}$
- $\phi(z) = \dfrac{z^n + 8}{z^n - 1}$



- Cupolae
- Rotation Group Z_n
- $\beta(z) = \dfrac{27\,(z^n - 1)^4\,(3\,z^{2n} - 16\,z^n + 1728)^3}{4\,z^n\,(5\,z^n - 54)^3\,(9\,z^n + 40)^4}$

- $\beta'(w) = \dfrac{4w^4(w^2 - 20w + 105)^3}{(7w - 48)^3(3w - 32)^2(5w + 12)}$
- $\phi(z) = \dfrac{96z^n - 96}{9z^n + 40}$



- Elongated Pyramids
- Rotation Group Z_n
- $\beta(z) = 4\,(-665857 + 470832\,\sqrt{2}) \cdot \dfrac{z^n\,(z^n - 1)^4\left[z^n - 4\,(41 + 29\,\sqrt{2})\right]^3}{\left[(-24 + 17\,\sqrt{2})\,z^n + 1\right]^4\left[4\,(2 + \sqrt{2})\,z^n + 1\right]^3}$

- $\beta'(z) = \dfrac{4\,(835 + 872\,\sqrt{2})\,w^4\,(w - 1)^3\left[(11 + 8\sqrt{2})\,w + 1\right]}{\left[(8 + 9\sqrt{2})\,w + 1\right]^3\left[(8 - 5\sqrt{2})\,w - 1\right]}$
- $\phi(z) = \dfrac{z^n - 1}{(8 - 5\sqrt{2})\,z^n + (11 + 8\sqrt{2})}$

Discussion

- **Can we find the Belyi maps of all the Johnson solids via their symmetry group G?**
  This approach should work for most of the Johnson solids. Some of the Johnson solids have a complexe engineering, which makes it hard to determine the solid's corresponding hypermap. The hypermap is the most essential part of this approach. Once the hypermap is determined, the reduced-hypermap can be easily deduced and associated with a Belyi map. The difficulty of the Belyi maps' computation depend on the computational resources available.
- **Zvonkin and Magot's approach**
  They approach consist of finding Belyi maps f for the 7 geometric operations on the platonic solids, then compose them with the platonic solids' Belyi maps g to get the Archimedean and Catalan solids' Belyi maps $\beta = f \circ g$. It is very likely that this approach is unfeasible with the Johnson solids. This is the case because the 6 operations (associated with the Johnson solids) have geometric actions that are really intricate to model algebraically.
- **Other classes of solids as Dessins d'Enfants ?**
  The Kepler-Poinsot solids and Stellated solids. [4] [5]

References

[1] Magot, Nicolas, and Alexander Zvonkin. 'Belyi functions for Archimedean solids.' Discrete Mathematics 217, no. 1 (2000): 249-271.

[2] Zvonkin, Alexander K. 'Belyi functions: examples, properties, and applications.' In Proceedings AAECC, vol. 11, pp. 161-180. 2008.

[3] Weisstein, Eric W. 'Johnson Solid.' From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/JohnsonSolid.html.

[4] Weisstein, Eric W. 'Kepler-Poinsot Solid.' From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/Kepler-PoinsotSolid.html

[5] Weisstein, Eric W. 'Stellation.' From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/Stellation.html.

Acknowledgements

We would like to say Thank you to:
The National Science Foundation (NSF) and
Professor Steve Bell,
Dr. Joel Spira,
Andris "Andy" Zoltners,
Professor Edray Goins,
Matt Toeniskoetter.

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